NEW YORK UNIVERSITY
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MATHEMATICS RESEARCH GROUP

RESEARCH REPORT No. CX-3



# A RIGOROUS SOLUTION OF A MANY-BODY PROBLEM

by

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CONTRACT No. AF-19(122)-463

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The research reported in this document has been made possible through support and sponsorship extended by the Geophysics Research Division of the Air Force Cambridge Research Center, under Contract No. AF-19(122)-463. It is published for technical information only, and does not necessarily represent recommendations or conclusions of the sponsoring agency.

August, 1952

Manufactured in the United States of America
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of
New York University

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## ABSTRACT

The scattering cross-section is calculated for the problem of a particle incident on another bound to the origin. The scattered particle interacts both with the fixed potential at the origin and the bound particle. Two solutions are obtained: one for the case where the two particles are similar and one for the case where they are dissimilar. The motion of each particle is one-dimensional, and the interactions are artificially chosen so that the Schrödinger equation is integrable. Thus, although the model chosen here has no physical reality, it is expected that the results will be useful in evaluating approximation techniques for physical particle scattering problems.

## I. Introduction

For the investigation of the dissociation and recombination processes in the ionosphere it is necessary to find approximation methods which give accurate results for small energies of the scattering particles. In order to test the effectiveness of, for instance, the Born approximation method or the Schwinger variational method, it is useful to consider examples of many-body problems for which rigorous solutions can be found. One can compare the results obtained by the various approximation methods with the exact solutions and gain some insight into the value of these methods for various energy ranges. We shall discuss two such examples. In the first example the particles are different from each other and their description is asymmetrical. In the second example the particles are identical and exchange effects are important.

## II. Different Particles

We shall consider the following one dimensional two-body problem: Particle 2 is bound to the origin by a δ-function potential. Particle 1 is incident on the origin from the left with momentum $K_o$. Particle 1 interacts with particle 2 only when both the particles are at the origin. The Schrödinger equation which describes this problem is

$$\left\{ \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - 2B\delta(x_2) + A\delta(x_1)\delta(x_2) \right\} \psi(x_1 x_2) = E\psi(x_1 x_2). \tag{1}$$

In Equation (1), the mass of the particles is taken to be dimensionless, and the momentum has the dimension of an inverse length. The magnitude of the masses of particles 1 and 2 is chosen to be 1/2. A and B are constants.

From (1) we see that particles 1 do not interact at the origin, whereas particles 2 do interact. Particle 1 therefore cannot be bound at the origin, and hence those exchange phenomena cannot arise in which particle 1 is captured at the origin and particle 2 is ionized.

In order to solve (1) it is expedient[1] to transform into momentum space. The wave functions in momentum space are much better suited for the calculation of the different cross-sections than the wave functions in coordinate space.

If one inserts the Fourier transform[†]

$$\psi(x_1 x_2) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(k_1, k_2)\, e^{i(k_1 x_1 + k_2 x_2)}\, dk_1 dk_2 \tag{2}$$

---

† The factor $1/2\pi$ is necessary in order that the wave function be properly normalized in coordinate space when it is normalized in momentum space.

in (1), one obtains the following integral equation for $f(k_1,k_2)$:

$$(k_1^2 + k_2^2 - E) f(k_1,k_2) - \frac{B}{\pi}\int_{-\infty}^{+\infty} f(k_1 k_2)\,dk_2 + \frac{A}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(k_1,k_2)\,dk_1\,dk_2 = 0. \quad (3)$$

The last integral in (3) diverges. In physical terms, the interaction $A\delta(x_1)\delta(x_2)$ is so singular that it produces no scattered wave. There are two ways to avoid this difficulty. First we can carry out a limiting process in momentum space in which the coupling constant A goes to zero as the integration boundaries go to infinity. Second, we can introduce into momentum space more general hermitian interactions which from the start will lead to convergent results and which nevertheless permit the exact integration of the Schrödinger equation.

The first method is very impractical if one wishes to assess the accuracy of the various approximation methods of quantum mechanics by comparing the results derived by these methods with the rigorous solutions of our examples. For instance, the solution of the Schrödinger equation in the Born approximation cannot be found here in a simple way. We therefore turn to the second method. The following more general interaction, for instance, fulfills the two requirements mentioned above (convergent results and exact integration of the Schrödinger equation):

$$(k_1' k_2' \,|V_w|\, k_1 k_2) = g(k_1' k_2')\, g^*(k_1 k_2),$$

where $g(k_1 k_2)$ must approach zero at least as rapidly as $\frac{1}{\sqrt{k}}$ when either $k_1 \rightarrow \infty$ or $k_2 \rightarrow \infty$.

This interaction no longer represents a diagonal matrix in coordinate space. However, nothing is fundamentally altered in the behavior of the asymptotic scattered wave, which we need for calculating the cross-section; nor does the interaction affect the various approximation methods for the solution of the Schrödinger equation. The hermitian properties of the interaction preserve the continuity equation and insure that the eigenvalues of the Schrödinger equation are real. Here we will not concern ourselves further with the physical implications of these more general interactions, since for our purposes they are necessary only in obtaining finite results.

For our asymmetrical example now we choose the following interaction:

$$(k_1'k_2' \,|V_w|\, k_1k_2) = \frac{A}{4\pi^2} \cdot \frac{1}{k_1'+iT} \; \frac{1}{k_1-iT} \,. \tag{4}$$

The Schrödinger equation (3) will thereby be altered in the following way:

$$(k_1'^2+k_2'^2-E)f(k_1',k_2') - \frac{B}{\pi}\int f(k_1',k_2')dk_2 + \frac{A}{4\pi^2}\,\frac{1}{k_1'+iT}\iint\frac{1}{k_1-iT}f(k_1,k_2)dk_1dk_2 = 0. \tag{5}$$

There are two methods of solving (5). The first is a direct solution; one sets

$$\iint \frac{f(k_1,k_2)}{k_1-iT}\, dk_1dk_2 = C$$

and

$$\int f(k_1,k_2)dk_2 = D(k_1),$$

and then solves for $f(k_1,k_2)$. C and D are subsequently determined by reinsertion of the solution thus obtained into (5). The second method consists of expanding $f(k_1,k_2)$ in a series of the complete set of eigenfunctions of the unperturbed Schrödinger equation (5) (A=0).

We wish to discuss both methods here and to show how they both lead to the same solution. This is important, since in the second example, in which both particles are identical, one can only use the second method of solution.

The first method of solving (5) is as follows:

Since particle 2 is bound to the origin before the impact, while particle 1 is incident from the left with impulse $K_o$, we set

$$f(k_1',k_2') = \delta(k_1'-K_o)\,\frac{N}{k_2'^2+B^2} + f_1(k_1',k_2')$$
$$N = \sqrt{\frac{2}{\pi}}\; B^{\frac{3}{2}} \,. \tag{6}$$

$\frac{N}{k_2'^2+B^2}$ is the eigenfunction normalized to 1 which describes the binding of particles to the origin with binding energy $B^2$. $f_1(k_1',k_2')$ describes the scattered wave which arises from the interaction of the two particles at the origin. It must contain the elastically scattered wave as well as the inelastically scattered one, in which particle 2 is no longer bound (there is only one bound state in this problem).

From (5) one obtains the following equation for $f_1(k_1',k_2')$:

$$f_1(k_1',k_2') = \frac{1}{k_1'^2+k_2'^2-E}\left\{\frac{B}{\pi}D_1(k_1') - \frac{A}{4\pi^2}\frac{C}{k_1'+iT}\right\} \tag{7}$$

$$E = K_o^2 - B^2 .$$

Since $f_1(k_1',k_2')$ represents a scattered wave, it must contain only outgoing waves when represented in momentum space. One can satisfy this requirement in the usual way by adding to E a small positive imaginary part which one allows to approach zero after all the integrations have been done.

By inserting (7) in the equation which determines $D_1(k_1')$ one obtains

$$D_1(k_1') = \frac{iAC}{4\pi(k_1'+iT)}\ \frac{1}{Bi-\sqrt{E-k_1'^2}} .$$

Then

$$f_1(k_1',k_2') = \frac{AC\cdot\sqrt{E-k_1'^2}}{4\pi^2(k_1'^2+k_2'^2-E)(k_1'+iT)(Bi-\sqrt{E-k_1'^2})} . \tag{8}$$

We get C from the equation

$$C = \iint \frac{1}{k_1-iT}\left\{f_o(k_1,k_2)+f_1(k_1,k_2)\right\}dk_1dk_2 . \tag{9}$$

In order to determine the cross-section for the elastic scattering of particle 1 and for the inelastic scattering in which particle 2 is ionized, one requires, as is well-known, only that portion of the wave function (8) which determines the asymptotic behavior of the wave function in coordinate space. This means that in momentum space one need only know the behavior of the wave function in the immediate neighborhood of its poles on the real axis, since this behavior determines the asymptotic behavior of the wave function. The remainder of the wave function determines the behavior of the particles when they are close together.

One can infer from the preceding remarks and from (8) that

$$\begin{aligned} f_1(k_1,k_2)_{\text{asympt.}} &= \frac{AC}{4\pi^2}\ \frac{-2B^2}{(k_1+iT)N}\ \frac{N}{k_2^2+B^2}\ \frac{1}{k_1^2-K_o^2} \\ &+ \frac{AC}{4\pi^2}\ \frac{|k_2|}{(k_1+iT)(Bi-|k_2|)}\ \frac{1}{k_1^2+k_2^2-E} . \end{aligned} \tag{10}$$

The first term in (10) suffices to determine the cross-section for elastic scattering of particle 1, while the second term determines the cross-section for

inelastic scattering. In order to find the cross-section for the inelastic scattering, we rewrite the second term in (10) in the coordinate space of particle 1:

$$f_1(x_1,k_2) = \frac{AC}{4\pi^2\sqrt{2\pi}} \frac{\pi i}{\sqrt{E-k_2^2}} \left\{ \frac{|k_2|}{(Bi-|k_2|)(\sqrt{E-k_2^2}+iT)} e^{i\sqrt{E-k_2^2}\, x_1} \right.$$

$$\left. - \frac{|k_2|}{(Bi-|k_2|)(-\sqrt{E-k_2^2}+iT)} e^{-i\sqrt{E-k_2^2}\, x_1} \right\} \qquad (11)$$

$$= ae^{i\sqrt{E-k_2^2}\, x_1} + be^{-i\sqrt{E-k_2^2}\, x_1} .$$

$$x_1 > 0 \qquad\qquad x_1 < 0$$

The current density of particle 1 with momentum $|k_1|$ or $-|k_1|$ where $|k_1| = |E-k_2^2|$, when the corresponding momentum of the ionized particle 2 lies between $k_2$ and $(k_2+dk_2)$, is given by

$$\frac{|k_1|}{m_1} a^*a\, dk_2$$

and

$$b^*b \frac{|k_1|}{m_1} dk_2$$

respectively. The current density of the incoming particle 1 is $\frac{K_o}{2\pi m}$, as one can see by rewriting $\delta(k_1-K_o)$ in coordinate space. The inelastic scattering cross-section of particle 1 on particle 2 is then given by

$$Q_{12} = \frac{2\pi}{K_o} \int_{-\sqrt{E}}^{+\sqrt{E}} (a^*a + b^*b)\, |k_1|\, dk_2 . \qquad (12)$$

The limits of integration are $-\sqrt{E}$ and $+\sqrt{E}$ since the ionized particle 2 can have at most the energy of the incoming particle 1 minus its own binding energy.

When we now consider the first term in (10), we note that the factor $N/k_2^2+B^2$ describes the normalized eigenfunction of a particle 2 bound at the origin. The transformation of this term in the coordinate space of particle 1 gives

$$\bar{f}_1(x_1 k_2) = -\frac{AC}{4\pi^2}\,\frac{\pi i 2B^2}{K_o\sqrt{2\pi}\,N}\left\{\underset{x_1>0}{\frac{1}{K_o+iT}\,e^{iK_o x_1}}+\underset{x_1<0}{\frac{1}{-K_o+iT}\,e^{-iK_o x_1}}\right\}\cdot\frac{N}{k_2^2+B^2}$$

$$= (\underset{x_1>0}{\bar{a}e^{iK_o x_1}} + \underset{x_1<0}{\bar{b}e^{-iK_o x_1}})\,\frac{N}{k_2^2+B^2}\ . \tag{13}$$

With this we get for the cross-section of elastic scattering of particle 1

$$Q_{el} = \frac{2\pi}{K_o}\,(\bar{a}\,\bar{a}^* + \bar{b}\,\bar{b}^*)K_o\ . \tag{14}$$

For further examination of the various approximation processes, it is necessary to determine the ratio of the intensity of the scattered wave to the intensity of the incident wave. This ratio can best be determined from the individual scattering cross-sections. We will therefore write down explicit expressions for the elastic and inelastic scattering cross-sections. The evaluation of the formulas (9),(12), (14) gives us

$$Q_{el} = \frac{A^2C\ C^*B}{4\pi K_o^2(K_o^2+T^2)}$$

$$Q_{ion} = \frac{1}{K_o}\,\frac{A^2CC^*}{8\pi}\,\frac{1}{E+B^2+T^2}\left\{\frac{\sqrt{E+T^2}}{T}-\frac{B}{\sqrt{E+B^2}}\right\}, \tag{15}$$

where

$$C=\frac{N\pi}{B(K_o-iT)}\ \frac{1}{1-\frac{A}{4(K_o^2+T^2)}\left[\frac{B}{T}+\frac{\sqrt{K_o^2-B^2+T^2}}{\pi T}\,\ln\frac{\sqrt{K_o^2-B^2+T^2}-T}{\sqrt{K_o^2-B^2+T^2}+T}-\frac{B}{\pi K_o}\,\ln\frac{K_o-B}{K_o+B}-i\left(\frac{B}{K_o}+\frac{\sqrt{K_o^2-B^2+T^2}}{T}\right)\right]}$$

$$N = \sqrt{\frac{2}{\pi}}\,B^{\frac{3}{2}}$$

$$E = K_o^2 - B^2$$

$$\mathrm{Im}\ K_o = \mathrm{Im}\ E > 0\ .$$

For the numerical evaluation of the quantities C, $Q_{el}$, and $Q_{ion}$, it is important to have a relation at hand which we can use to check the numerical calculations. One relation of this kind is the continuity equation which states that the number of particles 1 incident per unit of time must be equal to the number of particles 1 scattered per unit of time. Using equations (11), (13) and (15) we obtain the following:

$$\frac{A}{4\pi}\,\frac{2B^2 i}{(K_o^2+T^2)\Pi K_o}\left[(K_o-iT)\,C-(K_o+iT)C^*\right]=Q_{el}+Q_{ion},\quad K_o^2>B^2\,. \tag{16}$$

If $K_o^2 < B^2$, then $Q_{ion} = 0$. This means that for $K_o^2 < B^2$, the continuity equation (16) is altered radically. However, this alteration is compensated for by the fact that when $K_o = B$, C possesses a logarithmic branch point which, when $K_o^2 < B^2$, produces an additional imaginary term in the denominator of C:

$$i\,\frac{A}{4(K_o^2+T^2)}\left(\frac{B}{K_o}-\frac{\sqrt{K_o^2-B^2+T^2}}{T}\right)\quad .$$

If $T^2 < B^2$, then C has another branch-point for $K_o^2+T^2 = B^2$, but this branch-point does not produce an additional imaginary term.

Now we come to the second method for solving (5).

As was indicated previously, we will expand $f(k_1,k_2)$ in a series of eigenfunctions of the unperturbed Schrödinger equation (3), with A = 0. These eigenfunctions are:

$$(K_1K_2|k_1k_2)=\left\{\delta(k_2-K_2)+\frac{|K_2|}{|K_2|-Bi}\,\frac{B}{\pi}\,\frac{1}{k_2^2-K_2^2-i\alpha}\right\}\delta(k_1-K_1)\qquad \alpha>0 \tag{16a}$$

$$(K_1 2_B|k_1k_2)=\frac{N}{k_2^2+B^2}\quad\delta(k_1-K_1). \tag{16b}$$

The eigenfunctions (16a) describe a particle coming in with impulse $K_1$ and a particle 2 coming in with impulse $K_2$ being scattered at the origin. The eigenfunctions (16b) describe an incoming particle 1 with impulse $K_1$ and a particle 2 bound to the origin.

The eigenfunctions (16) are so normalized that

$$\iint(K_1K_2|k_1k_2)(K_1'K_2'|k_1k_2)^*dk_1dk_2 = \delta(K_1-K_1')\delta(K_2-K_2')$$

and (17)

$$\iint(K_1 2_B|k_1k_2)(K_1'2_B|k_1k_2)^*\,dk_1dk_2 = \delta(K_1-K_1').$$

One can then make an expansion of $f_1(k_1,k_2)$ in terms of these eigenfunctions:

$$f_1(k_1,k_2)=\iint f_1(K_1,K_2)dK_1dK_2(K_1K_2|k_1k_2)+\int f_1(K_1)dK_1(K_1 2_B|k_1k_2). \tag{18}$$

If one inserts this into (3) and again puts

$$f(k_1,k_2)=\delta(k_1-K_0)\,\frac{N}{k_2^2+B^2}\;+f_1(k_1,k_2),$$

one obtains

$$\iint (K_1^2+K_2^2-E) f_1(K_1,K_2) dK_1 dK_2 (K_1K_2|k_1k_2)$$

$$+ \int (K_1^2-B^2-E) f_1(K_1) dK_1 (K_1 2_B|k_1k_2) = - \frac{AC}{4\pi^2(k_1+iT)} . \tag{19a}$$

Here, just as in the first method,

$$C = \left\{ \iint \frac{\delta(k_1-K_1)}{k_1-iT} \quad \frac{N}{k_2^2+B^2} \quad dk_1 dk_2 + \iint \frac{f_1(k_1,k_2)}{k_1-iT} \quad dk_1 dk_2 . \right. \tag{19b}$$

If one multiplies (19a) with the complex conjugate of (16a) and (16b) in turn, and then integrates over $k_1$ and $k_2$, one obtains the following for $f_1(K_1,K_2)$ and $f_1(K_1)$:

$$\begin{aligned} f_1(K_1,K_2) &= - \frac{AC}{4\pi^2(K_1^2+K_2^2-E)} \iint \frac{(K_1K_2|k_1k_2)^*}{(k_1+iT)} \quad dk_1 dk_2 \\ &= - \frac{AC}{4\pi^2(K_1^2+K_2^2-E)} \quad \frac{|K_2|}{|K_2|+iB} \cdot \frac{1}{K_1+iT} \end{aligned} \tag{20a}$$

$$\begin{aligned} f_1(K_1) &= \frac{-AC}{4\pi^2(K_1^2-B^2-E)} \iint \frac{(K_1 2_B|k_1k_2)^*}{(k_1+iT)} \quad dk_1 dk_2 \\ &= - \frac{ACN}{4\pi . B . (K_1^2-B^2-E)} \cdot \frac{1}{K_1+iT} . \end{aligned} \tag{20b}$$

In order to demonstrate that $f_1(K_1,K_2)$ and $f_1(K_1)$ describe outgoing waves in coordinate space one must add a small positive imaginary part to E.

If one transforms to momentum space using equation (15), the integrals become rather complicated, since one cannot integrate in the complex plane because (20a) is not an analytic function of the variable $K_2$. On the other hand, the integration for the asymptotic behavior in coordinate space is simple, since only the singularities on the real axis are important. An explicit evaluation, of course, gives the same result.

The calculation of the asymptotic parts gives:

$$\begin{aligned} \underset{\text{asympt}}{f_1(k_1,k_2)} &= - \frac{ACN}{4\pi B(k_1^2-K_o^2)} \quad \frac{1}{k_1+iT} \quad \frac{N}{k_2^2+B^2} \\ &+ \frac{AC}{4\pi^2} \quad \frac{|k_2|}{(k_1+iT)(Bi-|k_2|)} \quad \frac{1}{k_1^2+k_2^2-E} . \end{aligned} \tag{21}$$

Comparing (21) and (10), one sees that both methods give the same results provided

we get the same result for C by both methods. With the help of equation (18), (19b) and (2) we can easily prove that this is the case

We now come to our second example of many-body problems.

## III Identical particles

For this more complicated problem we start with the following Schrödinger equation:

$$\left\{-\frac{\partial^2}{\partial x_1^2}-\frac{\partial^2}{\partial x_2^2}-2B\delta(x_1)-2B\delta(x_2)+A\delta(x_1)\cdot\delta(x_2)\right\}\psi(x_1x_2)=E\,\psi(x_1x_2)\,. \qquad (22)$$

Equation (22) differs from the original Schrödinger equation (1) only insofar as both particle 1 and 2 are in direct interaction at the origin. This new Schrödinger equation is therefore completely symmetric in the coordinates of particles 1 and 2 and therefore this equation can describe the two particle as indistinguishable. On the other hand this direct interaction of particle 1 at the origin makes it impossible to solve equation (22) by the first method discussed in section II. Therefore we have to use the second method.

Transforming again to momentum space, equation (22) becomes:

$$(k_1^2+k_2^2-E)f(k_1,k_2)-\frac{B}{\pi}\int f(k_1,k_2)dk_1-\frac{B}{\pi}\int f(k_1,k_2)dk_2+\frac{A}{4\pi^2}\iint f(k_1,k_2)dk_1dk_2=0\,. \qquad (23)$$

In the last integral of (23) the same divergence difficulties appear as in the preceding example. Therefore we choose now the following hermitian interaction potential in momentum space between the particles 1 and 2 and the origin; this interaction is symmetrical with respect to the two particles and avoids all divergence difficulties:

$$(k_1'k_2'|V_w|k_1k_2)=\frac{A}{4\pi^2}\frac{1}{k_1'^2+T^2}\,\frac{1}{k_2'^2+T^2}\,\frac{1}{k_1^2+T^2}\,\frac{1}{k_2^2+T^2}\,. \qquad (24)$$

The Schrödinger equation (23) will thereby be altered in the following way:

$$(k_1'^2+k_2'^2-E)f(k_1,k_2)-\frac{B}{\pi}\int f(k_1,k_2')dk_1-\frac{B}{\pi}\int f(k_1',k_2)dk_2$$

$$+\frac{A}{4\pi^2}\frac{1}{k_1'^2+T^2}\,\frac{1}{k_2'^2+T^2}\iint\frac{f(k_1,k_2)dk_1dk_2}{(k_1^2+T^2)(k_2^2+T^2)}=0. \qquad (25)$$

For the solution of (25) we set

$$f(k_1,k_2)=f_o(k_1,k_2)+f_1(k_1,k_2) \qquad (26a)$$

with

$$f_o(k_1,k_2) = \left\{ \delta(k_1-K_o) + \frac{K_o}{K_o-Bi}\ \frac{B}{\pi}\frac{1}{k_1^2-K_o^2} \right\} \frac{N}{k_2^2+B^2} \tag{26b}$$

$$K_o > 0 \ ; \ N = \sqrt{\frac{2}{\pi}}\ B^{\frac{3}{2}} \ ,$$

as we did in solving (6). Here $f_o(k_1,k_2)$ is a solution of (25) if $A = 0$, and it describes particle 1 with momentum $K_o$ incident from $-\infty$ and scattered by interaction at the origin. The scattered wave is described by the second term in the brackets of (26b). Particle 2 is bound to the origin and is described by the function $\frac{N}{k_2^2+B^2}$. We now wish to obtain from $f(k_1,k_2)$ the physically meaningful symmetrical and antisymmetrical solutions of (25) which describe the elastic or inelastic scattering of a particle with momentum $K_o$ incident from $-\infty$ by the origin. Another particle, indistinguishable from the first, is bound at the origin before the scattering. We have the following two solutions:

$$\underset{\text{symm.}}{f(k_1,k_2)} = \frac{1}{\sqrt{2}}\left\{ f(k_1,k_2) + f(k_2,k_1) \right\} \tag{26c}$$

$$\underset{\text{antisymm.}}{f(k_1,k_2)} = \frac{1}{\sqrt{2}}\left\{ f(k_1,k_2) - f(k_2,k_1) \right\} . \tag{26d}$$

In order that the wavefunction for scattering in coordinate space describe outgoing waves, it is necessary again to add a small imaginary part to E.

When A is not zero, an additional scattered wave arises from the interaction of both particles at the origin. In (26a) this additional scattered wave is designated by $f_1(k_1,k_2)$. In order to find this scattered wave, it is necessary, according to the second method, to expand $f_1(k_1,k_2)$ in terms of the eigenfunctions of equation (25), with $A = 0$. Therefore (in analogy with previous definitions) we have:

$$\begin{aligned}(K_1K_2|k_1k_2) = &\left\{ \delta(k_1-K_1) + \frac{|K_1|}{|K_1|-iB}\frac{B}{\pi}\frac{1}{k_1^2-K_1^2-i\alpha} \right\} \\ &\times \left\{ \delta(k_2-K_2) + \frac{|K_2|}{|K_2|-iB}\frac{B}{\pi}\frac{1}{k_2^2-K_2^2-i\alpha} \right\} ;\end{aligned} \tag{27a}$$

$$(K_1 2_B|k_1k_2) = \left\{ \delta(k_1-K_1) + \frac{|K_1|}{|K_1|-iB}\frac{B}{\pi}\frac{1}{k_1^2-K_1^2-i\alpha} \right\} \frac{N}{k_2^2+B^2} \ ; \tag{27b}$$

$$(1_B K_2|k_1k_2) = \left\{ \delta(k_2-K_2) + \frac{|K_2|}{|K_2|-iB}\,\frac{B}{\pi}\,\frac{1}{k_2^2-K_2^2-i\alpha} \right\} \frac{N}{k_1^2+B^2}\ ; \tag{27c}$$

$$(1_B 2_B|k_1k_2) = \frac{N}{k_1^2+B^2} \cdot \frac{N}{k_2^2+B^2}\ . \tag{27d}$$

In the eigenfunctions (27a) of (25) (A=0), particle 1 with impulse $K_1$ and particle 2 with impulse $K_2$ are incident and are scattered by the origin. Equations (27b) and (27c) describe elastic scattering of particles 1 and 2. When particle 1 is incoming and scattered elastically at the origin, particle 2 is bound there, and vice versa. In (27d) both particles are bound at the origin. The eigenfunctions (27) are so normalized that

$$\begin{aligned}
&\iint (K_1K_2|k_1k_2)(K_1'K_2'|k_1k_2)^* dk_1dk_2 = \delta(K_1-K_1')\delta(K_2-K_2')\\
&\iint (K_1 2_B|k_1k_2)(K_1' 2_B|k_1k_2)^* dk_1dk_2 = \delta(K_1-K_1')\\
&\iint (1_B K_2|k_1k_2)(1_B K_2'|k_1k_2)^* dk_1dk_2 = \delta(K_2-K_2')\\
&\iint (1_B 2_B|k_1k_2)(1_B 2_B|k_1k_2)^* dk_1dk_2 = 1.
\end{aligned} \tag{28}$$

If, as in example 1, we set

$$\begin{aligned}
f_1(k_1,k_2) = &\underbrace{\iint f_1(K_1,K_2)dK_1dK_2(K_1K_2|k_1k_2)}_{F(k_1,k_2)} + \underbrace{\int f_1(K_1)dK_1(K_1 2_B|k_1k_2)}_{G(k_1k_2)}\\
&+ \underbrace{\int f_1(K_2)dK_2(1_B K_2|k_1k_2)}_{H(k_1k_2)} + \underbrace{(1,2)_B(1_B 2_B|k_1k_2)}_{Im(k_1k_2)}
\end{aligned} \tag{29}$$

and insert this in (25), we obtain with the help of (27)

$$\begin{aligned}
&\iint (K_1^2+K_2^2-E)f_1(K_1,K_2)dK_1dK_2(K_1K_2|k_1k_2) + \int (K_1^2-B^2-E)\,f_1(K_1)dK_1(K_1 2_B|k_1k_2)\\
&\qquad + \int (-B^2+K_2^2-E)f_1(K_2)\,dK_2(1_B K_2|k_1k_2) + (-2B^2-E)(1,2)_B(1_B 2_B|k_1k_2) \qquad (30a)\\
&\qquad = -\frac{A}{4\pi^2}\,\frac{M}{(k_1^2+T^2)(k_2^2+T^2)}
\end{aligned}$$

with

$$M = \iint \frac{\{f_o(k_1,k_2)+f_1(k_1,k_2)\}}{(k_1^2+T^2)(k_2^2+T^2)}\ dk_1dk_2\ . \tag{30b}$$

If one multiplies (30a) with each part of the complex conjugate of (27) in succession and integrates, one obtains the following for $f_1(K_1,K_2)$, $f_1(K_1)$, $f_1(K_2)$, and $(1,2)_B$:

$$f_1(K_1K_2) = -\frac{AM}{4\pi^2T^2}\frac{|K_1|\cdot|K_2|(T-B)^2}{(K_1^2+K_2^2-E)}\frac{1}{(K_1^2+T^2)}\frac{1}{(K_2^2+T^2)}\frac{1}{|K_1|+iB}\frac{1}{|K_2|+iB}$$

$$f_1(K_1) = -\frac{AM}{4\pi\cdot T^2\cdot B}\cdot\frac{|K_1|(T-B)N}{(K_1^2-B^2-E)(T+B)}\frac{1}{K_1^2+T^2}\frac{1}{|K_1|+iB}$$

$$f_1(K_2) = -\frac{AM}{4\pi^2T^2\cdot B}\frac{|K_2|(T-B)N}{(K_2^2-B^2-E)(T+B)}\frac{1}{K_2^2+T^2}\frac{1}{|K_2|+iB} \tag{31}$$

$$(1,2)_B = \frac{AM\,N^2}{4(2B^2+E)(T+B)^2\cdot T^2\cdot B^2}\;.$$

We shall consider first the symmetrical solution of (25). If we insert (31) in (29) and use (26c), we get the following for the asymptotic part of $f_{\text{symm.}}(k_1,k_2)$:

$$\begin{aligned}
f_{\text{symm. asympt.}}(k_1,k_2) &= f_{o\,\text{symm.asympt.}}(k_1,k_2) + f_{1\,\text{symm.asympt.}}(k_1,k_2)\\
&= \frac{1}{\sqrt{2}}\left\{\delta(k_1-K_o)\frac{N}{k_2^2+B^2}+\delta(k_2-K_o)\frac{N}{k_1^2+B^2}\right\}\\
&+\left\{\left(\frac{1}{\sqrt{2}}\frac{K_o}{K_o-iB}\frac{B}{\pi}-\frac{AM\sqrt{2}\,N(T-B)K_o}{4\pi BT^2(K_o^2+T^2)(T+B)(K_o-iB)}\right)\left(\frac{1}{(k_1^2-K_o^2)}\frac{N}{(k_2^2+B^2)}+\frac{1}{k_2^2-K_o^2}\frac{N}{k_1^2+B^2}\right)\right\}\\
&-\frac{AM\sqrt{2}}{4\pi^2T^2}\frac{(T-B)^2|k_1||k_2|}{(k_1^2+T^2)(k_2^2+T^2)(|k_1|-iB)(|k_2|-iB)}\frac{1}{k_1^2+k_2^2-E}\;,\quad K_o>0.
\end{aligned} \tag{32}$$

The expression in the first curly brackets in (32) represents the incident wave of the two particles without scattering. The expression in the second curly brackets represents the elastic scattering of the two particles at the origin. The term with the factor AM represents the additional scattering which occurs because the particles can interact at the origin.

From (32) we obtain the following cross-sections for the elastic scattering and the ionization scattering:

$$Q_{el} = \frac{4}{E^2+2B^2}\left\{\frac{B}{\sqrt{2}} - \frac{AM\sqrt{2}}{4BT^2}\,\frac{(T-B)N}{(E+B^2+T^2)(T+B)}\right\}\left\{\frac{B}{\sqrt{2}} - \frac{AM^*\sqrt{2}(T-B)N}{4BT^2(E+B^2+T^2)(T+B)}\right\}$$

$$Q_{ion} = \frac{A^2MM^*\cdot 2}{8\pi\sqrt{E+B^2}\,T^4}\left[\frac{1}{\sqrt{E+T^2}(2T^2+E)^2(E+T^2+B^2)(T^2-B^2)}\left\{2T(E+T^2)\,\frac{6T^4+6ET^2-2EB^2-2B^4+E^2}{(E+2T^2)(E+T^2+B^2)(T^2-B^2)} - \frac{E+2T^2}{T}\right\} - \frac{2B\sqrt{E+B^2}}{(T^2-B^2)^2(E+T^2+B^2)^2(E+2B^2)}\right], \qquad E = K_o^2 - B^2. \tag{33}$$

We get M from (26a), (26b), (29), (30b) and (31):

$$M = \frac{S}{1-AR}$$

with

$$S = \pi\cdot\frac{N\sqrt{E+B^2}\,(T-B)}{T^2B(T+B)(E+B^2+T^2)(\sqrt{E+B^2}-iB)} \tag{34}$$

and

$$R = -\frac{(T-B)^2}{4T^4(T+B)^2}\left[\left\{\frac{2}{(E+2T^2)(E+T^2+B^2)\cdot\pi}\left[\frac{E^2(T^2-B^2)}{T\sqrt{E+T^2}(E+2T^2)^2} - \frac{2T\sqrt{E+T^2}}{(E+T^2+B^2)}\,\ln\frac{\sqrt{T^2+E}-T}{\sqrt{E}}\right.\right.\right.$$

$$\left. + \frac{2\sqrt{E+B^2}}{(E+2B^2)}\,\frac{B(E+2T^2)}{(E+T^2+B^2)}\,\ln\frac{\sqrt{E+B^2}-B}{\sqrt{E}} - \frac{(T^2-B^2)}{(E+2T^2)}\right] - B\,\frac{EB^2+ET^2+B^4+3T^4}{T(T^2-B^2)^2(E+T^2+B^2)^2}$$

$$\left. - \frac{(4T^4+EB^2+ET^2)(B^2+T^2)}{4T^2(T^2-B^2)^2(E+2T^2)^2} - \frac{B^2}{(T^2-B^2)^2(E+2B^2)} + \frac{2T^2B^2-ET^2-EB^2-6T^4}{2(E+2T^2)^3(T^2-B^2)}\right\}$$

$$+ i\,\frac{1}{(E+2T^2)(E+T^2+B^2)}\left\{\frac{(T^2-B^2)(2T^4+3ET^2-E^2)}{2T\sqrt{E+T^2}(E+2T^2)^2} + \frac{\sqrt{E+T^2}(7T^4+B^4+3ET^2+EB^2)}{2T(E+2T^2)(E+B^2+T^2)}\right.$$

$$\left.\left. + \frac{\sqrt{E+B^2}\cdot 2B(E+2T^2)}{(E+2B^2)(E+T^2+B^2)}\right\}\right].$$

We cannot, as before, simply use the continuity equation for one of the kinds of particles to check the numerical calculations, since not only particle 1 but also particle 2 is incident. But we may obtain an analogous relation when we consider the four possibilities: each incoming particle (1 or 2) may (a) go through unscattered, (b) be scattered elastically, (c) be exchanged with the bound particle, or (d) be scattered inelastically. Making use of (33) this relationship is written as follows:

$$\sqrt{2}\left\{\frac{2B^2}{\sqrt{2}(E+2B^2)}+\frac{iAN(T-B)}{4T^2B(E+T^2+B^2)(T+B)}\left(\frac{M}{\sqrt{E+B^2-iB}}-\frac{M^*}{\sqrt{E+B^2+iB}}\right)\right\}=Q_{el}+Q_{ion}\,, \tag{35}$$

In the symmetrical solution considered here, the mutual interaction of the particles causes an increase in the amplitude of the scattering wave. This increase has two components, the exchange scattering wave and the usual scattering wave. From (32) we can see that the amplitudes of these waves are equal. If we formulate the antisymmetrical solution of (25) with the help of (26d), (29), (30b), and (31), we see that for the antisymmetrical solution no additional scattering of the two particles comes about through their mutual interaction. This is not a general result but follows from the particular form of the interaction we chose .

That the scattering wave of the symmetrical solution is larger than the scattering wave of the antisymmetrical solution is physically understandable, because the antisymmetrical solution implies that both particles cannot be at the origin at the same time, in contrast to the behavior of the particles in the symmetrical solution.

## IV Conclusion

In conclusion we shall make some brief remarks on the Born and the Oppenheimer approximation methods in view of our example 2.

Using the Born method to solve (23) approximately, one considers A as a small magnitude and expands $f(k_1,k_2)$ in terms of the eigenfunctions of the simplified Schrödinger equation (23), $(A = 0)$. One then obtains $f(k_1,k_2)$ as power series expansion in A. This is almost the same method as we used above for the treatment of the rigorous solution. The only difference is that we now obtain the constant M (see Equation (34)) as a power series expansion in A. From this one can readily see that this approximation method will work well as long as

$$\left|\iint(k_1k_2|V_w|k_1'k_2')f_1(k_1',k_2')dk_1'dk_2'\right| \ll \left|\iint(k_1k_2|V_w|k_1'k_2')f_0(k_1',k_2')dk_1'dk_2'\right|. \tag{36}$$

In general, an approximation of this sort works only when one can regard one particle in a three body problem as infinitely heavy. Only then are the eigenfunctions of the simplified Schrödinger equation products of two-particle eigenfunctions: that is, we can carry out a separation of variables. However, when all three particles have finite masses (as is the case, for instance, with neutron and proton scattering with the deuteron[2]), even the simplified Schrödinger equation describes a real three-body problem because then already in the 0-th approximation forces will be indirectly transmitted over the third particle between the other particles. The mathematical implication of this is that now one cannot carry out a separation of variables for the solution of the simplified Schrödinger equation. Therefore one must alter the method just discussed for such cases. One way of doing this is to consider the

interaction of particle 1 at the origin as small in, for instance, our example, and not only that of both particles. However, this means that the description of the two particles is no longer symmetrical. Also, with this method we can no longer take into account the exchange scattering, namely the fact that the incident and the bound particles change places and the previously bound particle flies off freely. The reason for this difficulty is actually not the asymmetrical description of the two particles, but the fact that we regard the interaction of particle 1 at the origin as small. In order to take this into account, using our example 2, we will call the coupling constant which belongs to this interaction $\bar{B}$. We now obtain the solution $f(k_1,k_2)$ from (23) as a power series expansion $f(k_1,k_2)$ in the coupling constants A and $\bar{B}$. From (32) we can see that the situation — particle 2 with impulse $\pm K_o = \pm\sqrt{E+\bar{B}^2}$ is scattered, and particle 1 remains bound at the origin — is described in the following part of the rigorous eigenfunction (32), by the following term:

$$\frac{1}{k_2^2-(E+B^2)}\;\frac{1}{k_1^2+k_2^2-E}\;.$$

For $K_2^2 = K_o^2+B^2$,

$$\frac{1}{k_1^2+k_2^2-E} = \frac{1}{k_1^2+B^2}\;. \tag{37}$$

In $\bar{f}(k_1,k_2)$, this part turns up as a power series expansion to $\bar{B}$, which diverges for $|k_2| \rightarrow \sqrt{E+\bar{B}^2}$, just in the range which is of interest for the exchange scattering. From this we can see clearly why this method cannot describe exchange scattering. Also it becomes evident that the same difficulty is encountered when this method is applied to exchange of two distinguishable particles. (For a fuller discussion of this point see reference 1.)

In order to describe the particles in a symmetrical formulation and to be able to include exchange scattering in the equation, one can expand $f(k_1,k_2)$ in terms of a non-orthogonal system of functions, similar to what Oppenheimer has done[3]. In this system of functions we consider first particle 1 as free and particle 2 as interacting with particle 3; then we consider particle 2 as free and particle 1 as interacting with the third particle. As one can see, this procedure is similar to the Heitler-London method for the calculation of the hydrogen molecule.

The exact investigation of this procedure using the second example presented in this paper will be undertaken and will be written in a future research report.

Acknowledgements

The author has profited from discussion with Prof. Sidney Borowitz, Dr. Harry Moses, and Mr. Solomon Schwebel, all at New York University.

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